RAPID COMMUNICATIONS

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Singularity spectrum of self-organized criticality

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I introduce a simple continuous probability theory based on the Ginzburg-Landau equation that provides a common analytical basis to relate and describe the main features of two seemingly different phenomena of condensed-matter physics, namely self-organized criticality and multifractality.

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The concept of self-organized criticality (SOC) [1] has attracted great interest recently, both analytically [2, 3] and experimentally [4, 5]. The idea behind SOC is that a certain class of dynamical many-body systems drive themselves into a statistically stationary critical state, with no intrinsic length or time scale, where they exhibit fractal behavior and generate $1/f$ noise. Besides SOC, the generalization of fractal growth to self-similar multifractals has also attracted considerable attention over the past years [6, 7]. Theoretical models to describe multifractality (MF) have been concerned with mean-field arguments [8] and standard renormalization-group methods [9].

Motivated by the suggestion that SOC supports the appearance of fractal structures [1], it is natural to ask then if there is a common principle underlying the seemingly unrelated phenomena of SOC and MF. As far as I know a fixed scale transformation method [10], developed for fractal growth, has been used to investigate analytically the nature of two-dimensional (2D) clusters in SOC. Henceforth, it is also tempting to search for a unifying scenario that underpins a plausible link between MF and SOC. In fact this is the motivation for this work in which I only take a step in that direction.

In this Rapid Communication I propose a simple continuous probability theory based on the Ginzburg-Landau (GL) equation [11] that combines together the concepts of SOC and MF. In this goal I explore an analytical basis which allows one to unravel the genesis of power-law correlations in space from the point of view of a nonlinear singularity spectrum equivalent to multifractals and to obtain further insight into the physics governing this crossover.

A crucial feature of the present formalism is to consider that *all random variables* in a 1D space, $\mathbb{R}^1$, are functions of the coordinate variable χ which I map into an equivalent independent variable ζ—say (energy)/(unit force)—characterizing a random system. Then, all probabilities may became expressible in terms of the uniform *probability distribution function*

$$\mathcal{G}(\zeta_2) - \mathcal{G}(\zeta_1) = \mathcal{P}\{\zeta_1 < \zeta \leq \zeta_2\} \approx \int_{\zeta_1}^{\zeta_2} \phi(\zeta)\, d\zeta \quad , \qquad (1)$$

where $\{\,\}$ indicates the function interval and ϕ is a uniform *probability density* on the line (or $\mathbb{R}^1$) which needs to be specified. Within this continuous probability model I assume $\phi(\zeta) \equiv \frac{\phi_0}{2}\{1 + \mu H(\zeta)\}$, such that $\phi(\zeta \to +\infty)/\phi_0 \to 0$ and $\phi(\zeta \to -\infty)/\phi_0 \to 1$. I postulate $H(\zeta)$ to be given by the real solutions of the static, dimensionless GL-like equation: $\partial^2 H(\zeta^*)/\partial(\zeta^*)^2 + pH(\zeta^*) - qH^3(\zeta^*) = 0, \quad \zeta^* \varepsilon D$, where $\zeta^* \equiv \zeta/\zeta_0$, $[p,q] > 0$ and ζ_0 is a coefficient of dim[length]. If the 1D domain D is infinite, then the GL posseses the stable, kink solution: $H(\zeta^*) = \pm\sqrt{\frac{p}{q}}\, \tanh(\zeta^* \sqrt{\frac{p}{2}})$. Using this result it is possible to establish a relation for the probability distribution $\mathcal{P}$ as follows.

The integral of Eq. (1) over the limits: $\zeta_2 \equiv \lambda_1\zeta_0 \geq \zeta + \lambda_2\zeta_0 \equiv \zeta_1$, is

$$\mathcal{G}(\lambda_1\zeta_0) - \mathcal{G}(\zeta + \lambda_2\zeta_0) = \frac{\phi_0}{2} \int_{\zeta+\lambda_2\zeta_0}^{\lambda_1\zeta_0} \left\{ 1 \pm \mu\, \tanh\left(\frac{\zeta'}{\zeta_0}\right)\right\} d\zeta' \equiv -\tau(\zeta) \quad , \qquad (2)$$

which defines the function $\tau(\zeta)$. Therein I set $p = q = 2$ to reduce the free parameters. These integration limits lead to the condition

$$\lambda_2 - \lambda_1 + \zeta^* \leq 0 \quad . \qquad (3)$$

The sign in Eq. (2) implies that the $\mathcal{G}$ functions satisfy $\mathcal{G}(\zeta + \lambda_2\zeta_0) > \mathcal{G}(\lambda_1\zeta_0)$ for $\zeta \neq 0$, which throughout the theory are undefined, whereas λ_i (i=1,2) restrict the range of ζ^*.

Suppose $\lambda_2 \neq \lambda_1$, then the above integral gives

$$\tau(\zeta^*) \approx \left(1 + \frac{\zeta^*}{\lambda_2 - \lambda_1}\right) \{\tau(0) \mp \mu\phi_0^* \ln\cosh\lambda_2\} \pm\mu\phi_0^* \left(\frac{\zeta^*}{\lambda_2 - \lambda_1} \ln\cosh\lambda_1 + \ln\cosh(\lambda_2 + \zeta^*)\right) \quad , \qquad (4)$$

in which $\phi_0^* \equiv \zeta_0\phi_0/2$ and $\tau(0) \equiv \mathcal{G}(\lambda_2\zeta_0) - \mathcal{G}(\lambda_1\zeta_0) = (\lambda_2 - \lambda_1)\phi_0^*\{1 \mp \mu\Gamma_\lambda\}$, such that $\Gamma_\lambda \equiv \frac{\ln\cosh\lambda_1 - \ln\cosh\lambda_2}{\lambda_2 - \lambda_1}$. When $\zeta^* = 0$, MF will be described assuming $\mathcal{G}(\lambda_2\zeta_0) < \mathcal{G}(\lambda_1\zeta_0)$, whereas for SOC: $\mathcal{G}(\lambda_2\zeta_0) > \mathcal{G}(\lambda_1\zeta_0)$.

To analyze the possible multifractal features of $\mathcal{P}$ I define: $\tau(\zeta^*) \equiv \{\lambda_2 - \lambda_1 + \zeta^*\}D_{\zeta^*}$. Then it follows that

$$D_{\zeta^*} \equiv \frac{1}{\lambda_2-\lambda_1}\{\tau(0) \mp \mu\phi_0^* \ln\cosh\lambda_2\} \pm \frac{\mu\phi_0^*}{\lambda_2-\lambda_1+\zeta^*}\left(\frac{\zeta^*}{\lambda_2-\lambda_1}\ln\cosh\lambda_1 + \ln\cosh(\lambda_2+\zeta^*)\right) \quad , \tag{5}$$

such that $\lambda_2 - \lambda_1 + \zeta^* \neq 0$. From this relation I obtain $D_{\zeta^*\to 0} = \frac{\tau(0)}{\lambda_2-\lambda_1}$, and $D_{\zeta^*\to+\infty} = D_{\zeta^*\to 0} \pm \mu\phi_0^*\{1+\Gamma_\lambda\}$; $D_{\zeta^*\to-\infty} = D_{\zeta^*\to 0} \mp \mu\phi_0^*\{1-\Gamma_\lambda\}$. However, if $\lambda_2-\lambda_1+\zeta^*=0$ then $D_{\zeta^*\to(\frac{-1}{\lambda_2-\lambda_1})} = D_{\zeta^*\to 0} \pm \mu\phi_0^*\{\Gamma_\lambda + \tanh\lambda_1\}$.

Complementary to τ I also define

$$\alpha(\zeta^*) \equiv \frac{\partial}{\partial\zeta^*}\tau(\zeta^*) = D_{\zeta^*\to 0} \pm \mu\phi_0^*\{\Gamma_\lambda + \tanh(\lambda_2+\zeta^*)\} \quad . \tag{6}$$

Therefore, it can be easily shown that $\alpha_{\max} \equiv \alpha(\zeta^* \to -\infty) = D_{\zeta^*\to-\infty}$, and $\alpha_{\min} \equiv \alpha(\zeta^* \to +\infty) = D_{\zeta^*\to+\infty}$.

To relate these equations to MF and SOC, I impose a condition for α and D. If $\zeta^* \to +\infty$, then $D_{\zeta^*\to+\infty}$ and $\alpha(\zeta^*)$ are allowed to take the values 0 or $2D_{\zeta^*\to 0}$, depending on the phenomena. The case $\alpha(\zeta^*\to+\infty) \approx 0$ and $D_{\zeta^*\to+\infty} \approx 0$ will be in correspondence with MF. To achieve this I approximate

$$\mp\mu\phi_0^* \approx \frac{\epsilon D_{\zeta^*\to 0}}{(1+\Gamma_\lambda)} \quad , \tag{7}$$

where the integer factor $\epsilon \equiv \pm 1$ distinguishes such two cases.

According to definitions used in MF [12], I also consider $f(\alpha) \equiv \zeta^*\alpha(\zeta^*) - \tau(\zeta^*)$, and $C_{\zeta^*} \equiv -\frac{\partial^2}{\partial(\zeta^*)^2}\tau(\zeta^*) = \mp\mu\phi_0^* \operatorname{sech}^2(\lambda_2+\zeta^*)$. When $f(\alpha)$ and D_{ζ^*} are smooth functions of α and ζ^*, then $f(\alpha)$ can be related to $\tau(\zeta^*)$ by a Legendre transformation [12], which reflects a connection with thermodynamics. After this physical meaning and assuming $\phi_0^* > 0$ as discussed below, I realize that $\mu \to \mp 1$ since the analogous "specific heat" is $C_{\zeta^*} \geq 0$. This, in turn, implies that the reduced order parameter $\phi(\zeta)$ in Eq. (1) can take the desired values 0 and 1 when $\zeta^* \to +\infty$ and $\zeta^* \to -\infty$, respectively.

Besides this, the choice of $\mu = \mp 1$ implies that I may also obtain Eq. (7) from $D_{\zeta^*\to 0}$ and $\tau(0)$, provided $\varepsilon \to +1$. However, if $\varepsilon \to -1$, then Eq. (7) is recovered by changing the sign of $\tau(0)$. This will became clear later on. The present theory is thus dependent on λ_i (i=1,2) and (the sign and magnitude of) $\tau(0)$, such that ζ satisfies the condition in Eq.(3) and ϕ_0^* is positive, satisfying Eq.(7).

(1) The case $\lambda_1 > \lambda_2$ *such that* $\lambda_2 > 0$. Figure 1(a) displays the dependence of τ on ζ^* for different values of λ_1 and λ_2, such that $\tau(0) = -1$ and $\varepsilon = 1$. Noting that $\lambda_1 - \lambda_2 = 1$ in all curves illustrated, so as to have $D_{\zeta^*\to 0} = 1$, then from Eq. (3) $\zeta \leq 1$. Hitherto, the present GL-based approch allows ζ^* to take on negative and positive values ≤ 1. However, in following plots concerning MF, the range of ζ^* is extended up to 3 for illustrative purposes. From Fig. 1(a) it can be seen that, on increasing the value of λ_1, there is a more rapid convergence of τ for positive ζ^* than for $\zeta^* < 0$; displaying thus typical features of MF [12]. Such a nontrivial behavior of τ illustrates the data collapse or breakdown of MF at $\zeta^* > 0$ where $\tau(\zeta^*) > 0$. This is in accordance with the MF structure of the function D_{ζ^*} shown in Fig. 2, which corresponds to a spectrum of fractal dimensions. I point out that D_{ζ^*} of Eq. (5) is not constant for positive or negative ζ^* so that it relates to a multifractal dimension. Hence, τ becomes also a nonlinear function of ζ^*. In fact this approach yields, e.g., for the full line in Fig. 1(a) the values $D_{\zeta^*\to 1} = 0.421$ and $D_{\zeta^*\to-\infty} = 3.53$.

In Fig. 3(a) I display α as a function of ζ^* for the same λ_1 and λ_2 as in Fig. 1(a). When $\lambda_1 = 1$ and $\lambda_2 = 0$ this function exhibits sharp variations around $\zeta^* = 0$ with a maximum value that shows a stronger dependence on λ_1 for positive ζ^* than for $\zeta^* < 0$. This rules out the possibility that the actual position for a critical value of ζ, at which the multifractal formalism actually breaks

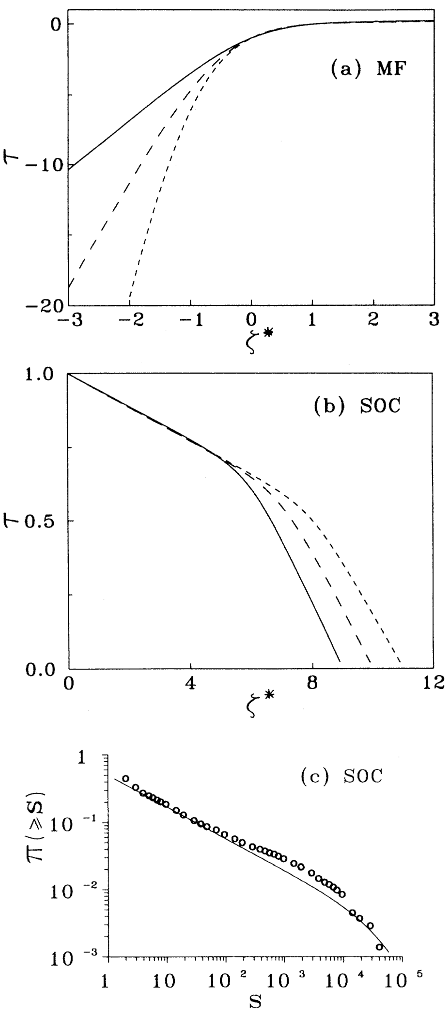

FIG. 1. (a) Analogous "free energy" τ vs ζ^* for MF using (full line) $\lambda_1 = 1$, $\lambda_2 = 0$; (– – – –) $\lambda_1 = 1.5$, λ_2=0.5; (· · · · · ·) $\lambda_1 = 2$, $\lambda_2 = 1$. (b) Normalized probability distribution function τ vs ζ^* for SOC using (full line) $\lambda_1 = 3$, $\lambda_2 = -6$; (– – – –) $\lambda_1 = 3$, $\lambda_2 = -7$; (· · · · · ·) $\lambda_1 = 3$, $\lambda_2 = -8$. (c) Present description of the probability $\pi(\geq s)$ for the SOC signal in a model of erosion [14].

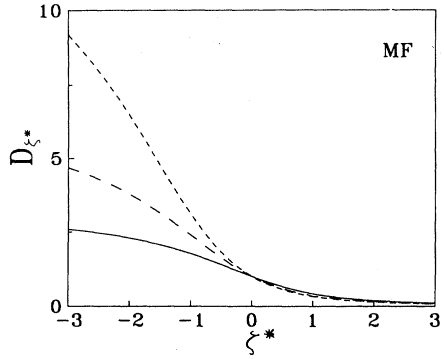

FIG. 2. Multifractal dimension D_{ζ^*} vs ζ^* using λ_1 and λ_2 as in 1(a).

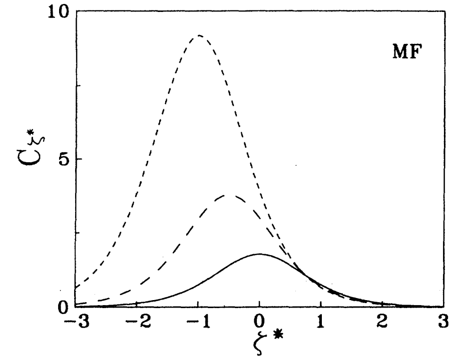

FIG. 4. Analogous "specific heat" C_{ζ^*} vs ζ^* using λ_1 and λ_2 as in 1(a).

down, is obtained at $\zeta_c^* = 0$. For other values of λ_1 and λ_2, ζ_c^* changes towards negative values. For $\zeta^* < \zeta_c^*$, $\tau(\zeta^*)$ is dominated by α which, in turn, varies with the magnitude of λ_2.

Characteristic features of a phase transition at ζ_c^* can be figured out by examining C_{ζ^*}, which is illustrated in Fig. 4. There is a sharp peak around the value $\zeta_c^* = 0$ for the case corresponding to the full line in Fig. 1(a). The heights and positions of these curves are strongly dependent on $\lambda_1 > \lambda_2$. Nicely, this finding is also similar to reported MF results [12]. The behavior of f against α for several values of λ_1 and λ_2 can be seen in Fig. 5(a). I find that, on increasing the magnitude of λ_1, the left-hand sides of these plots converge more rapidly than the right-hand sides which converge poorly. This is in complete agreement with the MF signal observed in the context of self-similar random resistor networks (open and full circles in Fig. 5(a) [13]) or, e.g., in diffusion-limited aggregation when λ_1 is related to the system size [7]. Given its interpretation of a multifractal character for fractal subsets, each with a different fractal dimension having singularity strength α, then $f(\alpha) \geq 0$.

All of these predictions resemble qualitatively the intriguing results observed in MF. In the present examples the maximum and minimum values of α can easily be estimated. This allows for the existence of a critical point

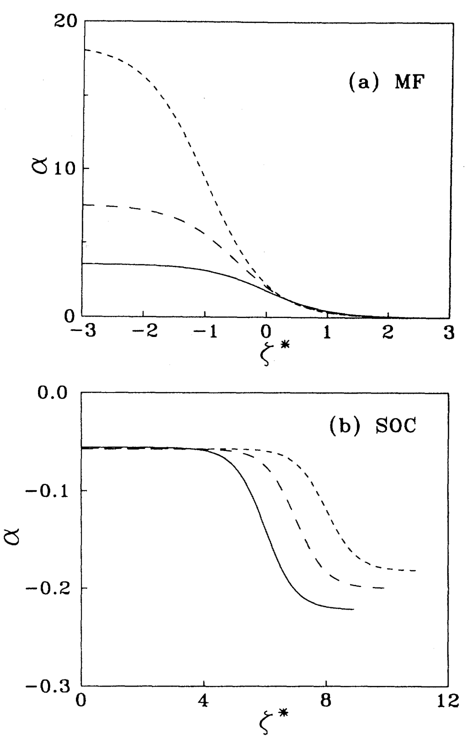

FIG. 3. (a) Analogous "internal energy" α vs ζ^* using λ_1 and λ_2 as in 1(a). (b) The function α of Eq.(6) vs ζ^* using λ_1 and λ_2 as in 1(b).

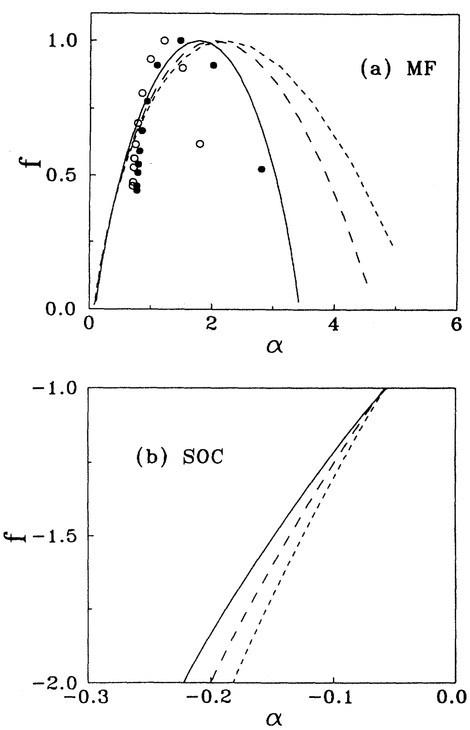

FIG. 5. (a) Analogous "entropy" f vs analogous "internal energy" α using λ_1 and λ_2 as in 1(a). Open and full circles are MF signals for self-similar random resistor networks [13]. (b) The nonlinear $f(\alpha)$ singularity spectrum proposed to characterize SOC using λ_1 and λ_2 as in 1(b).

ζ^* above which the infinite hierarchy of phases can be found, but below which a single phase appears characterized by $\alpha_{\rm max}$.

(2) The case $\lambda_1 > \lambda_2$ such that $\lambda_2 < 0$: In Fig. 1(b) the dependence of τ on ζ^* is plotted for different values of λ_1 and λ_2 such that, as a difference with MF, $\tau(0) = 1$ and $\varepsilon = -1$. This choice allows us to normalize $\tau(\zeta^*)$ and to mimic the main features of SOC, namely a power-law behavior, provided ζ^* is associated with the logarithmic function of some measured random event, say, s. As an illustrative example, in Fig. 1(c) I show a reasonable description of the probability π $(\geq s)$ of the SOC signal calculated within a model of erosion [14]. This demonstrates that the present theory does apply to self-organizing systems. For a class of continuous, cellular automaton models of earthquakes [15], τ can be reinterpreted as being the number of events with reduced released energy $E \sim e^{\zeta^*}$.

The theoretical curves in Fig. 1(b) refer to different values of λ_1 and λ_2. Using Eq. (3), the GL-based theory is valid for $\zeta \leq 9 - 11$, where I deal with $0 < \tau(\zeta^*) \leq 1$. A given slope of the linear behavior of the curves in this figure is determined by fixing $\lambda_1 > 0$. In particular, this parameter may be associated with the elastic parameter of the spring-block model for earthquakes [15]. The cutoff in the ζ^* axis may be related to the system size of cellular automaton modeling.

To see more clearly power-law features in the behavior of τ over a wide range of $\zeta^* > 0$ I investigate next the derivative of $\tau(\zeta^*)$, defined through α of Eq. (6), which is plotted in Fig. 3(b). The behavior of $\alpha(\zeta^*)$ indicates that for the smallest positive ζ^*, it converges to a constant negative value thus revealing the constant nature of the negative slopes in the $(\tau - \zeta^*)$ curves of Fig. 1(b). On increasing ζ^* each curve smoothly approaches a smaller value. Clearly, due to the probabilistic definition of τ such convergences of α need not to be considered and, hence, the relations between $\alpha_{\rm max,min}$ and $D_{\zeta^* \to \mp\infty}$ become meaningless for SOC.

After establishing this resemblance of a power-law description for τ, I continue applying anew the MF formalism to analyze SOC. In view of the features in $\alpha(\zeta^*)$ of Fig. 3(b), the second derivative of τ, i.e., C_{ζ^*} (not shown), presents a sharp peak around the inflection point of the function $\alpha(\zeta^*)$, say $\zeta^*_{\rm inf}$. As a difference to MF (c.f. Fig. 4), these peak heights reduce their magnitude on decreasing $\lambda_2 < \lambda_1$ and shift their position towards positive values of ζ^*. No phase transition as in the case of MF is expected because $\tau > 0$ restricts the range of valid $\zeta^* < \zeta^*_{\rm inf}$. Moreover, D_{ζ^*} for SOC (also not shown) is not constant on increasing ζ^* as in MF.

The singularity spectrum $f(\alpha)$ plays an alternative role when dealing with SOC as can be assessed from Fig. 5(b). In this plot $f(\alpha)$ exhibits a nonlinear behavior different from the parabolic shape of Fig. 5(a). In MF $f(\alpha)$ takes its maximum at $\alpha(\zeta^* = 0)$ whereas in SOC this spectrum becames a monotonically increasing (negative) function of (negative) α. On decreasing the magnitude of λ_2 the SOC curves converge to -1 and separate out as a function of decreasing α within the range of validity of α in Fig. 3(b). Since $\tau(\zeta^*)$ is positive, then $-2 < f(\alpha) < -1$. I suggest this new aspect of $f(\alpha)$ to be a fundamental property for the additional characterization of SOC. It is, therefore, most likely that the linear behavior displayed by $\tau(\zeta^*)$ [Fig. 1(b)], that is quantified via $\alpha(\zeta^*)$ [Fig. 3(b)], finds its root through the behavior of $f(\alpha)$ [Fig. 5(b)].

In conclusion, I have been able to shed light on a unifying formalism leading to both phenomena of MF and SOC using a continuous density probability $\phi(\zeta)$. This function has been related to $H(\zeta)$, which I postulated to be given by the real kink solutions of a dimensionless GL-like equation. Of course to use a continuous probability theory may be seen as being heuristic, especially so if simulations are done using discretized cell configurations. But, as I discussed, a great deal of relevant information can be extracted from a continuous approach which, essentially, does relay on the sign of λ_2, $\tau(0)$, and ε only. While the present (static) GL-based theory is extremely simple, it gives information about the complex origin of self-organized critical phenomena whose physics has been shown—to a good approximation—to be analogous to that required to describe MF. This theory also reflects the minimal ingredients that can give rise to an intrinsically critical state. Lastly, I recently learned that MF also emerges from GL equations with random initial conditions for its temporal evolution [11].

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